Differential Geometry—MTG 6256—Fall 1999
Problem Set 1: Fun with Matrices

Below, $M_n(\mathbf{R})$ denotes the vector space of $n \times n$ real matrices, and $\mathrm{GL}(n, \mathbf{R})$ the subset of invertible matrices. We give $M_n(\mathbf{R})$ the usual topology of $\mathbf{R}^{n^2}$.

The problems below are meant to be done in the given order; in many cases the results of earlier problems are applicable to later problems. When asked to find the derivative of a map, express your answer by writing down a formula that gives all directional derivatives.

1. (a) Let C be a fixed $n \times n$ matrix and define $l_C, r_C : M_n(\mathbf{R}) \to M_n(\mathbf{R})$ by $l_C(A) = CA, r_C(A) = AC$. Show that l_C and r_C are differentiable and find their derivatives.

(b) Define $\tau : M_n(\mathbf{R}) \to M_n(\mathbf{R})$ by $\tau(A) = A^t$ (the transpose of A). Show that τ is differentiable and find its derivative.

2. Let $g, h : M_n(\mathbf{R}) \to M_n(\mathbf{R})$ be differentiable and define $f(A) = g(A)h(A)$. Prove that f is differentiable, and express the derivative of f in terms of the derivatives of g and h.

3. Let $f(A) = A^t A$ for $A \in M_n(\mathbf{R})$. Show that f is differentiable and find its derivative.

4. Let $m \geq 1$ be an integer and let $f(A) = A^m$ for $A \in M_n(\mathbf{R})$. Show that f is differentiable and find its derivative.

5. Prove that $\mathrm{GL}(n, \mathbf{R})$ is an open, dense subset of $M_n(\mathbf{R})$.

6. Define $\iota : \mathrm{GL}(n, \mathbf{R}) \to M_n(\mathbf{R})$ by $\iota(A) = A^{-1}$. Show that ι is differentiable and find its derivative. (Before showing that ι is differentiable, I suggest you figure out what $D\iota$ would have to be—i.e. find the directional derivatives. Hint: problem 2.)

7. Extend the result of problem 5 to negative integral exponents. (For $A \in \mathrm{GL}(n, \mathbf{R})$ and $m \geq 1$, A^{-m} is defined to be $(A^{-1})^m$.)

8. The determinant function $\det : M_n(\mathbf{R}) \to \mathbf{R}$ is a polynomial in n^2 variables, so it is certainly C^1 (in fact C^∞). There are several ways to compute its derivative. The steps below constitute a method that involves little computation but a bit of thought.

(a) Let $I \in M_n(\mathbf{R})$ be the identity and let $B \in M_n(\mathbf{R})$. Compute $D_I(\det)(B)$.

(b) Let $A \in \mathrm{GL}(n, \mathbf{R}), B \in M_n(\mathbf{R})$. Compute $D_A(\det)(B)$. (Hint: use (a).) Re-express your result as a formula for the function $\log|\det|$.

(c) Use the density statement in problem 5 to extend the formula for $D_A(\det)$ from $A \in \mathrm{GL}(n, \mathbf{R})$ to $A \in M_n(\mathbf{R})$. The answer can be rewritten in terms of the "cofactor" matrix $\mathrm{cof}(A)$ that arises in computing the inverse of a matrix. (Recall that if A is invertible, then $A^{-1} = \frac{1}{\det(A)}\mathrm{cof}(A)$, or else the transpose of this, depending on your definition of $\mathrm{cof}(A)$.)