Differential Geometry—MTG 6256—Fall 1999
Problem Set 6

Throughout, M is a manifold of dimension n. $\Omega^*(M)$ denotes the exterior algebra of differential forms, i.e. $\sum_{p=0}^n \Omega^p(M)$ with wedge product. You may take "$\omega \in \Omega^*(M)$" to mean "$\omega \in \Omega^p(M)$ for some p" even though this is not what the notation literally means.

1. Let X be a vector field on M. Prove that ι_X is a graded derivation of $\Omega^*(M)$:

$$\iota_X(\omega \wedge \eta) = (\iota_X \omega) \wedge \eta + (-1)^{\deg(\omega)} \omega \wedge \iota_X \eta.$$

2. Let X be a vector field on M. Prove that for all $\omega \in \Omega^*(M)$,

$$\mathcal{L}_X \omega = d(\iota_X \omega) + \iota_X d\omega.$$

3. Let ω be a 2-form and let X, Y, Z be vector fields. Find the formula for $d\omega(X, Y, Z)$ analogous to the one derived in class (not involving local coordinates) for d of a 1-form.

4. In class we defined $\int_M \omega = \sum_\alpha \int_{U_{\beta(\alpha)}} \rho_\alpha \omega$, where $\omega \in \Omega^n(M)$, $\{U_\alpha\}$ is a cover of M coming from coordinate charts and $\{\rho_\alpha\}$ is a subordinate partition of unity, with $\mathrm{supp}(\rho_\alpha) \subset U_{\beta(\alpha)}$. Show that this definition is independent of the choice of partition of unity. (This problem can be done in just a few lines.)

5. Throughout this problem, let $\tilde{M}$ be a second manifold of dimension n and assume that $F : \tilde{M} \to M$ is a submersion. (For dimensional reasons, this implies that F_{*q} is an isomorphism at each $q \in \tilde{M}$.)

(a) Prove that for all $p \in M$, $F^{-1}(p)$ is a discrete set of points, and a finite set if $\tilde{M}$ is compact.

(b) Prove that if M is connected then the cardinality of $F^{-1}(p)$ is independent of p. If $\tilde{M}$ is compact, then this finite common value—the number of points in the pre-image of any $p \in M$—is called the *degree* of F.

(c) Prove that if $\tilde{M}$ is compact then F is a covering map. (A map $G : N \to M$, where N is a manifold, is a *covering map* if for all $p \in M$ there exists a neighborhood U for which $G^{-1}(U)$ is a disjoint union, possibly infinite, of sets V_i for which $G|_{V_i} : V_i \to U$ is a diffeomorphism. Examples: Regard S^1 as the unit circle in the complex plane. (i) For all integers $m \neq 0$, the map $z \mapsto z^m$ is a covering map of degree m. (ii) The map $t \mapsto e^{it}$ is a covering map from $\mathbf{R}$ to S^1.)

(d) Show that an orientation of M induces, via F, an orientation on $\tilde{M}$. (Hence if M is orientable, so is $\tilde{M}$.)

(e) Assume that $\tilde{M}$ is compact and that M is oriented; give $\tilde{M}$ the induced orientation. Show that for all $\omega \in \Omega^n(M)$,

$$\int_{\tilde{M}} F^*\omega = (\deg F) \int_M \omega$$