Differential Geometry—MTG 6256—Fall 1999
Problem Set 8

1. Let M be a manifold, E a vector bundle over M, F a sub-bundle of E, and h a metric on E. For all $p \in M$ define $F_p^\perp \subset E_p$ to be the h_p-orthogonal complement of F_p in E_p. Prove that this defines a smooth vector bundle $F^\perp$. (It is of course assumed that M, E, F and h are smooth, but smoothness of $F^\perp$ must be proven.)

2. Let M be a manifold and $N \subset M$ a closed submanifold.

(a) Let $f : N \to \mathbf{R}$ be a smooth function. Prove that there exists an open set $U \subset M$ containing N such that f extends to a smooth function $\tilde{f} U \to \mathbf{R}$ ("f extends to $\tilde{f}$" means that $\tilde{f}|_N = f$).

(b) Same as A, but start with a smooth function $f : N \to \mathbf{R}^k$, where k is arbitrary.

(c) Let E be a vector bundle over M, and suppose that s is a smooth section of $E|_N$. Prove that there exists an open set $U \subset M$ containing N such that s extends to a smooth section of $E|_U$. (This is a generalization of part (b), which can be viewed as the special case of (c) with $E = M \times \mathbf{R}^k$.)

Remark. If the submanifold N is assumed to be a closed subset of M, then in all three cases above the function or section smoothly extends to all of M, not just to a neighborhood U, but you're not being asked to prove this.

3. Let M be a manifold and E a vector bundle over M with metric h. Let ∇ be a connection on E that preserves h. Prove that A is the (matrix-valued) connection form of ∇ with respect to a local orthonormal basis of sections of E, then for all tangent vectors X, $A(X)$ is an *antisymmetric* matrix (equivalently, A is an antisymmetric matrix of 1-forms).

4. Let $\mathbf{R}^n$ have its standard metric and orientation. Let $\mu \in \Omega^n(\mathbf{R}^n)$ be the corresponding volume form on $\mathbf{R}^n$, and let $\omega \in \Omega^n(S^{n-1})$ be the induced volume form on S^{n-1} (where the orientation on the sphere is the one obtained by regarding the sphere as the boundary of the unit ball). Let $r : \mathbf{R}^n \to \mathbf{R}$ denote distance to the origin.

(a)Let $D^n(r) \subset \mathbf{R}^n$ be the disk of radius r, (i.e. $\{\mathbf{x} \in \mathbf{R}^n \mid \|\mathbf{x}\| \leq r\}$). Show that $\text{Volume}(D^n(r)) = r^n \text{Volume}(D^n(1))$.

(b) Compute the volume of $D^n(1)$ explicitly. (Using (a), and regarding $\mathbf{R}^{n+1}$ as $\mathbf{R}^n \times \mathbf{R}$, you should be able to do this by induction on n. Along the way you will need to compute a trigonometric integral I_n which can be expressed recursively in terms of I_{n-2}, so that after figuring out I_1 and I_2 you'll get a formula for I_n.)

(c) Let $\pi : \mathbf{R}^n - \{0\} \to S^{n-1}$ be radial projection; i.e. $\pi(\mathbf{x}) = \mathbf{x}/\|\mathbf{x}\| = \mathbf{x}/r$. Show that on the complement of the origin, $\mu = r^{n-1} dr \wedge \pi^* \omega$.

(d) Using (c) show that Volume$(S^{n-1}) = \frac{d}{dr}$Volume$(D^n(r))|_{r=1}$, and then use (a) and (b) to give a formula for Volume(S^n).

5. Let (M, g) be a Riemannian manifold, $\mathbf{g} : TM \to T^*M$ the isomorphism induced by g. The *gradient* of a function $f : M \to \mathbf{R}$ is the vector field $\mathbf{grad}(f) = \mathbf{g}^{-1}(df)$. Let $\{x^i\}$ be local coordinates on M. Express $\mathbf{grad}f$ in terms of $\{\partial f/\partial x^i\}$, the local basis $\partial/\partial x^i$ of TM, and the metric coefficients g_{ij} and/or g^{ij}.

6. Let (M, g) be a Riemannian manifold and let $\tilde{g}$ be a metric on M conformally related to g, so $\tilde{g} = e^{2\phi}g$ for some function $\phi : M \to \mathbf{R}$. Let $\nabla, \widetilde{\nabla}$ be the Levi-Civita connections of $g, \tilde{g}$ respectively. For all vector fields X, Y, compute $\widetilde{\nabla}_X Y$ in terms of $\nabla_X Y$ and $\mathbf{grad}(\phi)$.

7. Let S^2 and S^3 have their standard orientations and let $\alpha \in \Omega^2(S^2)$ satisfy $\int_{S^2} \alpha = 1$. Let $f : S^3 \to S^2$.

(a) Prove that $f^*\alpha \in \Omega^2(S^3)$ is an exact form.

(b) Assuming part (a), there exists a non-unique $\beta \in \Omega^1(S^3)$ such that $f^*\alpha = d\beta$. Define $h(f, \alpha, \beta) = \int_{S^3} f^*\alpha \wedge \beta$. Prove that $h(f, \alpha, \beta)$ is independent of the choice of β.

(c) Assuming (b), we can unambiguously define $h(f, \alpha)$ to be $h(f, \alpha, \beta)$ for any choice of β as above. Prove that $h(f, \alpha)$ is independent of the choice of α (as long as the total integral of α is 1). Thus we can unambiguously define $h(f)$ to be $h(f, \alpha)$ for any α. The number $h(f)$ is called the *Hopf invariant* of the map f.

(d) Recall that the *Hopf map* $f_1 : S^3 \to S^2$ is defined by regarding S^3 as the set $\{(z_1, z_2) \in \mathbf{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$, then mapping (z_1, z_2) to its equivalence class $[z_1, z_2] \in \mathbf{C}P^1$, then identifying $\mathbf{C}P^1$ with S^2 in a standard way. Tracing through this definition one finds that

$$f_1(z_1, z_2) = (2\mathrm{Re}(z_2\bar{z}_1), 2\mathrm{Im}(z_2\bar{z}_1), |z_2|^2 - |z_1|^2).$$

Compute the Hopf invariant of f_1.