Mathematical grammar and correct use of terminology

Sentences in mathematical writing often use mathematical symbols. These symbols have very precise meanings. Some examples are:

1. "=" stands for "equals", "which equals", or "is equal to". It does not stand for "Doing the next step in this problem, I arrive at the expression to the right of the equals sign."

2. "∀" stands for "for all", "for every", or "for each". **Note that each of these possibilities is a two-word phrase beginning with "for".** *The symbol "∀"* **NEVER** *stands for the single word "all", "every" or "each".*

3. "∃" stands for "there exists" or "there exist". *It* **never** *stands for "some" or "for some".*

4. " $\Longrightarrow$ " stands for "implies", "implying" or "which implies". *It* **NEVER** *stands for "then".* The symbol has the meaning "implying" or "which implies" only when it is immediately preceded by a comma.

   Note: The single-arrow symbol "→" is **not** an implication-arrow.

5. "$\Longleftarrow$" stands for "which is implied by" (this can also be read "implies", if you read from right to left or from the bottom of a page up).

6. " $\Longleftrightarrow$ " stands for "if and only if", or, when the symbol is immediately preceded by a comma, "which is equivalent to".

7. "∈" stands for "is in" or "is an element of" (as in a sentence ending ". . . then $w \in V$ [period]"); for "be in" or "be an element of" (as in the sentence "Let $w \in V$ [period]"); or for "in" (as in the sentence "Let $v \in V$ be a vector orthogonal to w [period]").

8. "⊆" (synonym: ⊂) stands for "is a subset of" (as in the sentence "Therefore $W \subseteq V$ [period]"); "be a subset of" (as in the sentence "Let $W \subseteq V$ [period]"); or "subset" or "in" (as in the sentence "Let $W \subseteq V$ be a subspace [period]"). A common convention among many mathematicians, illustrated in the last usage above, is that "subset" can be used as a preposition (a word like *in* or *on* that indicates the relation of two objects to one another), not just as a noun.

Using mathematical symbols does not relieve the writer of the responsibility to punctuate his or her sentences correctly; generally the symbols do not incorporate punctuation marks.[1] Your written work should have the property that, when the conventional English meanings of your symbols are substituted for the symbols themselves, the result is a collection of sentences with correct grammar and punctuation, with logical connections between the sentences. In particular this applies to equations, which are examples of sentences, and to strings of equations, which are often used as long sentences that detail the logical flow of an argument. A common way to achieve mathematical gibberish is simply to write equations down on a page, with no words connecting them to indicate their logical relation to one another. A string of equations, each of which is implied by the previous one, should be unambiguously readable as a grammatical English sentence (complete with punctuation), although sometimes a very long and tedious one. For example, suppose you are asked to do the following problem.

Show that if x is a positive real number and $x^2 + 2x - 3 = 0$, then $x = 1$.

Valid proof. Assume x is a positive real number and $x^2 + 2x - 3 = 0$. Then

$$\begin{array}{rrcl} & x^2 + 2x - 3 & = & 0, \\ \Longrightarrow & (x+3)(x-1) & = & 0, \\ \Longrightarrow & x + 3 = 0 & \text{or} & x - 1 = 0, \\ \Longrightarrow & x = -3 & \text{or} & x = 1. \end{array}$$

Since x is positive, $x \neq -3$, and therefore $x = 1$. ■

The equation part of this argument reads in English as "Then $x^2 + 2x - 3$ equals 0, which implies $(x+3)(x-1)$ equals 0, which implies $x+3$ equals 0 or $x-1$ equals 0, which implies x equals -3 or x equals 1".

"Gibberish" version of same argument.

$$\begin{array}{rcl} x^2 + 2x - 3 & = & 0 \\ (x+3)(x-1) & = & 0 \\ x + 3 = 0 & & x - 1 = 0 \\ x = -3 & & x = 1 \\ x = 1 & & \end{array}$$

This reads "$x^2 + 2x - 3$ equals 0 $(x+3)(x-1)$ equals 0 $x+3$ equals 0 $x-1$ equals 0 x equals -3 x equals 1 x equals 1". Which way makes more sense?

[1] There are some exceptions to this general rule, but we will not discuss those explicitly in this handout.

Some common mistakes.

All of the mistakes below were taken from students' homework in past MAS 4105 classes.

1. Mistaken order of quantifiers.

   The symbols "$\forall$" and "$\exists$" are called *quantifiers*, as are the words or phrases for which they stand. The order and location in which quantifiers appear makes a difference. For example, consider the four statements:

   (1) $\forall$ persons p $\exists$ a shirt s such that s fits p.

   (2) $\exists$ a shirt s such that $\forall$ persons p, s fits p.

   (3) s fits p, for some shirt s, for all persons p.

   (4) s fits p, for all persons p, for some shirt s.

   The first statement says that every person can find a shirt that fits. The second says that there's one magic shirt that fits every single person. The third and fourth are so ambiguous as to be incomprehensible.

2. Mistakenly using "$\implies$" in place of "then".

   Do not replace the construction "if ... then" with "if ... $\implies$".

   Example of correct usage. "If $x = 5$, then $x^2 = 25$."

   Example of correct usage. "$x = 5 \implies x^2 = 25$." This reads "$x = 5$ implies $x^2 = 25$" and has *exactly* the same meaning as "If $x = 5$ then $x^2 = 25$."

   Example of incorrect usage. "If $x = 5 \implies x^2 = 25$." This reads "If $x = 5$ implies $x^2 = 25$", which is not even a complete sentence. It also phrases an absolute truth ($5^2 = 25$) as part of a conditional statement.

3. Mistaken omission of set braces.

   Example of correct usage. "$\mathbf{R}^n = \{(a_1, \ldots, a_n) \mid \text{each } a_i \in \mathbf{R}\}$."

   Example of incorrect usage. "$\mathbf{R}^n = (a_1, \ldots, a_n)$, where each $a_i \in \mathbf{R}$."

4. Mistakenly using parentheses instead of curly braces.

   Parentheses are used to denote an ordered n-tuple, not a set. For example, if $v_1, \ldots, v_n$ are elements of a vector space V, it is correct to write "$v_1, \ldots, v_n \in V$" or "$\{v_1, \ldots, v_n\} \subseteq V$", but incorrect to write "$(v_1, \ldots, v_n) \in V$" or "$(v_1, \ldots, v_n) \subseteq V$".

5. Mistaken usage of "such that" or "so that"

   Such that or *be such that* means "having the property that" or "have the property that". For example "Let x be such that $f(x) = 5$" means "Let x have the property that $f(x) = 5$".

Such that is not used to indicate that something is implied by something else. For example, it is *wrong* to write "Let $2x = 10$, such that $x = 5$." What you can correctly write instead is "Let $2x = 10$, so that $x = 5$." In contrast to *such that*, the phrase *so that*, when preceded by a comma, always means that what's to the right of the comma is implied by what's to the left.

"So that" *can* be used to mean the same thing as "such that" in certain cases, but *never* if it is preceded by a comma. (MAS 4105 students: For a valid use of "so that" to mean "such that", see the beginning of the definition of vector space on p. 6 of Friedberg, Insel, and Spence, 5th ed. Had the authors preceded "so that" by a comma, the meaning would have been entirely different. Had they replaced "so that" by "such that", the meaning would have been the same.) Below are some more examples. (In my MAS 4105 classes, students should postpone reading the first two until after we've defined "basis of a vector space" (Section 1.6 of Friedberg, Insel, and Spence).

Example of incorrect usage. "Let $\{v_1, \ldots, v_n\}$ be a basis of $\mathbf{R}^n$, such that any $v \in \mathbf{R}^n$ is a linear combination of $v_1, \ldots, v_n$."

Example of correct usage. "Let $\{v_1, \ldots, v_n\}$ be a basis of $\mathbf{R}^n$, so that any $v \in \mathbf{R}^n$ is a linear combination of $v_1, \ldots, v_n$."

Example of correct usage. "Choose vectors v_1, v_2 such that $v_1 \perp v_2$."

6. Improper use of the word "let"

The grammatical construction "Let X be such-and-such" or "Let X have this or that property" means "*Assume* X is such-and-such" or "*Assume* X has this or that property."[2] It is generally used to fix notation at the start of an argument. It cannot be used if X is not a variable object (e.g. if $X = 5$), or if X has (in the context of that argument) already been assumed to be something else, or if X has already been assumed to have the indicated property, or if you are about to assume that X has properties that could conflict with those you've already assigned it.

Example of correct usage. "Let $x \in \mathbf{R}$ satisfy $x^2 = 25$."

Example of incorrect usage. "Let $x = 5$ satisfy $x^2 = 25$."

Example of correct usage. "Let $x \in \mathbf{R}$ be such that $x^2 = 25$."

Example of correct usage. "Let $x = 5$, so that $x^2 = 25$."

[2] "Assume" and "Let" actually have slight differences in nuance, but those differences are more subtle than the issues this handout is intended to address.

Further mistakes involving terminology
(mostly for MAS 4105 students, after the relevant terminology as been defined)

1. The word "equation"

   An equation is a sentence with an equals-sign in it. Expressions that lack "=" signs (e.g. "$x + 5$") are not equations. In particular:

   - A vector is not an equation.
   - A sum of vectors is not an equation.
   - A linear combination of vectors is not an equation.
   - An inequality is not an equation.

2. The word *solution*

   Equations and inequalities can have solutions (see the handout "What is a solution?"). However,

   - A vector does not have a solution.
   - A matrix does not have a solution.

3. Miscellaneous

   - In a general vector space, there is no such thing as division by a vector.
   - In a general vector space, there is no such thing as "1 over a vector".
   - You cannot add a scalar to a vector in $\mathbf{R}^n$ (unless $n = 1$).
   - The *components* of a vector in $\mathbf{R}^n$ are real numbers; they are not elements of $\mathbf{R}^n$ (unless $n = 1$).
   - A *vector* is not the same thing as a *vector space.* (A set of vectors does not equal a vector.)
   - A *set* of vectors is not the same thing as a *linear combination* of vectors.
   - A *basis* is not the same thing as the vector space it's a basis of.
   - A basis is not the same thing as a *linear combination* of vectors.
   - A basis is not the same thing as a *sum* of vectors.
   - A basis is not the same thing as a *solution set* of a system of linear equations.
   - A *set* of linear combinations of vectors is not the same thing as a linear combination of vectors. (A linear combination of vectors is a single vector, not an infinite set of vectors. A *set* of linear combinations of nonzero vectors, with arbitrary coefficients, is an infinite set of vectors.)

- The *number of elements in a vector space* is not the same thing as the *number of elements in a basis* of that space. (The vector space $\{0\}$ has exactly one element. Every other vector space has infinitely many elements. For $n, m > 0$, $\mathbf{R}^n$ has "just as many" elements as $\mathbf{R}^m$, no matter which of n, m is larger.)
- A plus sign does not mean the same thing as a comma, or as the word "and". Do not put plus signs between elements of a list of vectors unless you really mean to add the vectors.
- A matrix is not the same thing as a linear system of equations. (One can *associate* a matrix to a linear system of equations, but they are not the same animal.)